Formulation of turbulence mechanics

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This paper presents a setup of turbulence mechanics for averaged description of turbulence, founded on laws of momentum, moment of momentum, and energy, complemented by common rheological principles for formulating constitutive relations between generalized forces and generalized velocities of the description. A kinematical-geometrical principle is adopted to determine internal rotating degrees of freedom of turbulent media generated by the eddy structure of turbulent flow fields. The connection between the formulated mechanics and some models (as K-ε model), widely used in practical engineering flow calculations, is established. As an example, the formulated mechanics is applied to describe some classical flow patterns.

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I. INTRODUCTION

Turbulence mechanics is a theory of turbulence formulated in terms of average fields and founded on conservative laws of momentum, moment of momentum, and energy. The balance equations, expressing these laws, are closed by applying common rheological principles for formulating constitutive relations expressing generalized forces of the description through the corresponding generalized velocities and by adopting some specific assumptions which may vary on considerations such as physics encompassed in the flow, the level of accuracy, and so on.

The basic question to answer in formulating any mechanical description of turbulence lies in determination of degrees of freedom of turbulent motion. Starting from Richardson's turbulence understanding [1] with complementary remarks by Kolmogoroff [2] (together referred to as the RK conception) it is easy to conclude the independence of internal rotating degrees of freedom in turbulent medium (formed as the summary effect of rotation of hierarchy of eddies of different scales with a cascading mechanism for their generation) from the degrees of freedom of its translatory motion described in terms of average velocity field. Indeed, large-scale eddies in the hierarchy draw their energy from average flow and average angular velocity of their rotation determined by vorticity of the average velocity field, while the small-scale eddies are not oriented, i.e., average velocity of their rotation is zero. It is clear that in this situation the mean angular velocity of eddy rotations over all scales cannot be determined by the vorticity of the average velocity field unambiguously and must be treated as independent of the average velocity. (This corollary of RK conception was not noticed by Richardson and Kolmogoroff themselves.) As a consequence, the law of moment of momentum should form an indispensable component of any setup of mechanical description of turbulence.

The independence of rotating and translatory degrees of freedom of turbulent motion was first broached by Mattioli [3]. Mattioli's idea was vivified in 1970s as indicating to a possible field of applications of moment hydrodynamics [4–9]. Concrete attempts in this direction [10–12] were made by ascribing micromorphic properties to turbulent media appearing beyond the scope of classical field theories. Due to the latter, the formulated idea as well as the idea of Mattioli did not live up to expectations.

Nikolaevski [13] attempted to revive those ideas by associating internal rotating degrees of freedom in turbulent media with volume (coarse grain) averaging. Although this approach expresses the rotational degrees of freedom in a turbulent medium in terms of conventional characteristics of turbulent flow field, it binds the nontriviality of internal moments to the finiteness of the linear scale of the differential volume. This statement ascribes a subjective sense to the internal moments in turbulent media and differs from the conventional understanding of differential volume as a volume of infinitesimally small linear scale.

The essentiality of the internal rotating degrees of freedom of turbulent motion and the absence of their satisfactory determination led the author to a kinematical-geometrical principle of determination of characteristics of internal rotation in turbulent media [14,15]. Indisputable advantage of the proposed approach (discussed in detail in Secs. II and III stands in the connectedness of the internal rotating degrees of freedom with local measurable flow field parameters, turning the statements of the theory verifiable. Concerning the latter, the proposed theory differs essentially from the ones used in Refs. [10–12] and finds the need to vitalize the ideas of the 1960s which have been forgotten to a large extent and are not actively pursued in most of today's turbulence studies.

The closure problem for the formulated turbulence mechanics (Sec. IV) is solved within the common rheology used in the mechanics of continuums. It is shown that specific solutions of the closure problem by adopting ideas of K, K-ε, and K-ω models, widely used in the majority of applications, open the door to their significant generalization. As an example, in Sec. V the formulated mechanics is applied to the description of some classical flow patterns.

II. THE SETUP OF MECHANICAL DESCRIPTION OF TURBULENCE: EQUATIONS OF BALANCE

We start the setup of turbulence mechanics from the definition of density of the internal moment of momentum per

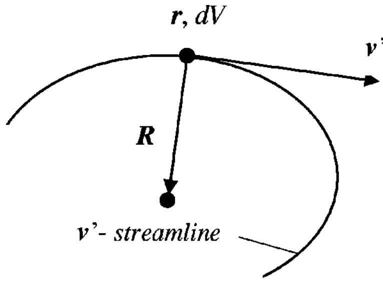

FIG. 1. The definition of **M** (1) at a fixed point determined by point vector **r**: unlike the conventional understanding of internal moments, defined as moments with respect to a fixed point inside differential volume dV associated with **r**, moment **M** at **r** defines as the average moment with respect to random centers of curvature (determined by **R**) outside the differential volume dV.

unit mass of turbulent flow field (henceforth the internal moment of momentum) in the form

$$\mathbf{M} = \langle \mathbf{v}' \times \mathbf{R} \rangle. \tag{1}$$

In Eq. (1) and thereafter the brackets denote statistical averaging (or an arbitrary averaging, satisfying the Reynolds averaging rules); $\mathbf{v}' = \mathbf{v} - \langle \mathbf{v} \rangle$ denotes the fluctuation (residual) constituent of the flow velocity $\mathbf{v}$; and $\mathbf{R} = \partial \mathbf{e}/\partial s |\partial \mathbf{e}/\partial s|^{-2}$, in which $\mathbf{e} = \mathbf{v}'/v'$ and s is length of the arc of $\mathbf{v}'$ streamline, which denotes the curvature radius of $\mathbf{v}'$ streamline (Fig. 1).

Let us note that **R**, contained in definition (1), can be extracted from the experimental data by using technique to measure Lagrangian velocities of tracer particles in turbulent flow [16] and from the data of direct numerical simulations [17]. This comment attributes **M** with measurability and all corollaries following from the definition of **M**, given by Eq. (1), with testability.

We also point out the following.

(i) **M** is defined for each point of the flow field and forms a continuum.

(ii) **M** (as a quantity determined on characteristics of fluctuating constituents of velocity field) is defined as independent of the average velocity $\langle \mathbf{v} \rangle$.

(iii) **M** is defined as a statistical characteristic of the motion field and cannot be interpreted (due to the randomness of **R**) as a moment with respect to any fixed moment center (in this sense **M** is similar to the spin in quantum mechanics).

(iv) Definition (1) is not related to the microproperties of the medium.

(v) In general

$$\mathbf{M} \neq 0 \tag{2}$$

[the property of turbulent medium, expressed by the condition (2), is called henceforth rotational anisotropy. It declares the existence of a preferred orientation of eddy rotations in turbulent medium].

(vi) The description of turbulent motions satisfying the condition (2) must be subjected to the laws of momentum and moment of momentum.

Let us list some other properties of turbulent continuum which follow from Eqs. (1) and (2).

(vii) The definition (1) suggests the definition of a kinematical characteristic of flow field $\mathbf{\Omega}$ corresponding to **M**, as

$$\mathbf{\Omega} = \left\langle \frac{\mathbf{v}' \times \mathbf{R}}{R^2} \right\rangle. \tag{3}$$

(viii) The density of turbulence energy per unit mass (henceforth the turbulence energy) $K^t = \frac{1}{2}\langle v'^2 \rangle$ decomposes into the sum of two constituents,

$$K_1^t = \tfrac{1}{2}\mathbf{M} \cdot \mathbf{\Omega}$$

and

$$K_2^t = \tfrac{1}{2}\langle \mathbf{M}' \cdot \mathbf{\Omega}' \rangle,$$

where

$$\mathbf{M}' = \mathbf{v}' \times \mathbf{R} - \mathbf{M}$$

and

$$\mathbf{\Omega}' = \frac{\mathbf{v}' \times \mathbf{R}}{R^2} - \mathbf{\Omega},$$

interpreted as the densities of energies of two-dimensional and of three-dimensional constituents of turbulence. [According to RK conception the energies K_1^t and K_2^t can be interpreted as energies of relatively large-scale (oriented) and relatively small-scale (nonoriented) constituents of turbulence. To avoid confusion with applying the terms “small-scale” and “large-scale” turbulence, we cease using them if either $K^t = K_1^t$ or $K^t = K_2^t$].

(ix) **M** and $\mathbf{\Omega}$ define the tensor of effective moment of inertia $\mathcal{J}$ for each flow field point, determined by

$$\mathbf{M} = \mathcal{J} \cdot \mathbf{\Omega}.$$

When $\mathcal{J}$ is isotropic ($\mathcal{J} = J \cdot \hat{\mathbf{1}}$, where $\hat{\mathbf{1}}$ is the unit tensor) it defines the parameter $l = \sqrt{J}$ with the dimension of length for every flow field point.

Here we point to the difference in physics of introduced $\mathbf{\Omega}$, defined by Eq. (3), and vorticity $\boldsymbol{\omega}$, defined by $\boldsymbol{\omega} = \nabla \times \langle \mathbf{v} \rangle / 2$. If $\boldsymbol{\omega}$ describes the angular velocity of rotation of a medium particle in differential volume dV surrounding the moment center and is determined by the average velocity field, then $\mathbf{\Omega}$ is determined as the average characteristic of fluctuating (residual) constituent of the velocity field. It expresses the average angular velocity of rotation of a medium particle in dV with respect to the momentary centers of curvature of $\mathbf{v}'$ streamlines. Quantities $\mathbf{\Omega}$ and $\boldsymbol{\omega}$, though different in their physical sense, have coinciding dimensions and may appear to have equal values. Definitions (1) and (3) disprove the idea according to which the independence of characteristics of internal rotation in turbulent media should follow only from averaging over some volume [13] since volume averaging and differentiation need not commute, i.e., may lead to $\langle \nabla \times \mathbf{v} \rangle \neq \nabla \times \langle \mathbf{v} \rangle$.

In addition to (i)–(ix), the definition (1) prescribes, together with Navier-Stokes equation and averaging rules, an algorithm for deriving the balance equations for momentum, moment of momentum ($\mathbf{M}$), and energy K_2^t, leading also to specific expressions of their terms. These equations follow as the averaged Navier-Stokes equation, as the averaged equation formed by multiplying the equation for $\mathbf{v}'$ (deduced as the difference between the Navier-Stokes equation and the averaged Navier-Stokes equation) from the right vectorially by $\mathbf{R}$, and as the difference of energy equations for K^t and K_1^t.

Below are the derived equations of balance:

$$\rho\frac{D}{Dt}\mathbf{u} = \{\sigma_{ij,j}\} + \rho\mathbf{F}, \tag{4}$$

$$\rho\frac{D}{Dt}\mathbf{M} = \{m_{kj,j}\} - \boldsymbol{\sigma} + \rho\mathbf{m}, \tag{5}$$

$$\rho\frac{D}{Dt}K_2^t = \boldsymbol{\nabla}\cdot\mathbf{h}_2^t - \psi + \Psi + B + \rho q. \tag{6}$$

In Eqs. (4)–(6), in addition to the notations explained above, $\mathbf{u}\equiv\langle\mathbf{v}\rangle$; ρ is the medium density (assumed to be constant); $D/Dt=\partial/\partial t+\mathbf{u}\cdot\boldsymbol{\nabla}$; $\mathbf{F}=\langle\mathbf{f}\rangle$, in which $\mathbf{f}$ denotes the density (per unit mass) of the nonaveraged body force acting on medium; σ_{ij} and m_{kj} denote the components of the stress tensor and the moment stress tensor, describing the diffusive transport of momentum and moment of momentum in a medium; $\boldsymbol{\sigma}=\{e_{kij}\sigma_{ij}\}$, where e_{kij} are the components of the Levi-Civitta tensor, denotes the dual vector of the antisymmetric constituent of the stress tensor; $\mathbf{m}$ denotes the density (per unit mass) of body moment acting on a medium; $\mathbf{h}_2^t$ denotes the diffusive flux vector for energy K_2^t; ψ describes the molecular dissipation of energy K_2^t; Ψ denotes the scattering function of energies $u^2/2$ and K_1^t into energy K_2^t; B denotes the term describing an additional mechanism of interaction between K_1^t and K_2^t differing from the one described by Ψ; and q denotes the term which describes internal source of K_2^t. In Eqs. (4)–(6) and henceforth, the index after comma denotes differentiation by the respective space coordinate, while the notations in braces denote the component representation of a tensor or vector quantity, wherein equivalent notation, arbitrary tensor or vector quantity ≡ {components of this quantity}, is used.

Consider now the case $\langle\mathbf{R}\rangle=0$. In this case the derivation of Eqs. (4)–(6) leads us to the following expressions for σ_{ij}, $\boldsymbol{\sigma}$, m_{kj}, $\mathbf{m}$, $\mathbf{h}_2^t$, ψ, Ψ, B, and q through characteristics of nonaveraged flow field:

$$\sigma_{ij} = \langle\sigma_{ij}^m\rangle + \sigma_{ij}^t,$$

where

$$\sigma_{ij}^t = -\rho\langle v_j' v_i'\rangle \tag{7}$$

are components of the turbulent stress tensor and $\sigma_{ij}^m=-p\delta_{ij}+2\mu^m v_{(i,j)}$ [p denotes thermodynamic pressure, δ_{ij} are components of unit tensor, μ^m is coefficient of molecular viscosity, and $v_{(i,j)}=\frac{1}{2}(v_{i,j}+v_{j,i})$] denote components of the molecular stress tensor;

$$\boldsymbol{\sigma} = -\rho\{e_{kis}\langle v_j' v_i' R_{s,j}\rangle\};$$

$$m_{kj} = -\rho\langle v_j' M_k'\rangle;$$

$\mathbf{m}=\mathbf{m}_f+\mathbf{m}_1+\mathbf{m}_2$, in which

$$\mathbf{m}_f = \langle\mathbf{f}'\times\mathbf{R}\rangle$$

($\mathbf{f}'=\mathbf{f}-\mathbf{F}$ denotes the fluctuating constituent of $\mathbf{f}$ field),

$$\mathbf{m}_1 = \left\langle\mathbf{v}'\times\frac{\partial}{\partial t}\mathbf{R}\right\rangle, \tag{8}$$

and

$$\mathbf{m}_2 = \{e_{kis}(\langle v_i' u_j R_{s,j}\rangle + \langle v_j' R_s\rangle u_{i,j})\}; \tag{9}$$

$$\mathbf{h}_2^t = \mathbf{h}^t - \mathbf{h}_1^t$$

in which

$$\mathbf{h}^t = \{-\rho\langle v_j' K^{v'}\rangle + \langle\sigma_{ij}^{m'} v_i'\rangle\}$$

(where $K^{v'}=v'^2/2$) and

$$\mathbf{h}_1^t = \{m_{kj}\Omega_k\};$$

$$\psi = \langle\sigma_{ij}^{m'} v_{i,j}'\rangle;$$

$$B = -\rho\mathbf{m}_2\cdot\boldsymbol{\Omega};$$

$$\Psi = \sum_{\alpha=1}^{4}\mathcal{F}_\alpha\mathcal{V}_\alpha, \tag{10}$$

in which $\mathcal{F}_\alpha$ and $\mathcal{V}_\alpha$ denote the generalized forces and velocities of the description, defined as

$$\mathcal{F}_\alpha = \{\sigma_{(ij)}^t, \boldsymbol{\sigma}, m_{ij}, -\rho\mathbf{m}_1\}$$

and

$$\mathcal{V}_\alpha = \{u_{(i,j)}, \boldsymbol{\Omega}-\boldsymbol{\omega}, \Omega_{i,j}, \boldsymbol{\Omega}\},$$

where $\sigma_{(ij)}^t=(\sigma_{ij}^t+\sigma_{ji}^t)/2$ is the symmetric part of the turbulent stress tensor; and

$$q = \langle(\mathbf{f}'\times\mathbf{R})'\cdot\boldsymbol{\Omega}'\rangle.$$

Let us list some remarks concerning the situation described by Eq. (4)–(6).

(i) No more assumptions besides the listed one in the section preceding Eqs. (4)–(6) are adopted in the deduction process.

(ii) The mechanics of turbulence based on Eqs. (4)–(6) does not reject classical theories and models (for instance, K model, K-ε model, and K-ω model), founded on the balance equation (4) and on the equation for full turbulence energy K^t, following from Eq. (6) for rotationally isotropic turbulence, but complements them.

(iii) The derivation of Eqs. (4)–(6) exhibits not only specific expressions for the terms of the equations but also the structure of the generalized forces and generalized velocities—the base of formulation of constitutive relations (Sec. III).

(iv) The mechanics based on Eqs. (4)–(6) declares asymmetry of the turbulent stress tensor. In a coordinate system associated with each flow field point (x_i) and with an origin placed at a random point with the coordinates x_i+R_i, for $\boldsymbol{\sigma}$ we have

$$\boldsymbol{\sigma} = -\rho\{e_{kij}\langle v_i' v_j'\rangle\}.$$

According to definition of $\mathbf{M}$, $\boldsymbol{\sigma}$ is interpreted as the moment acting on the internal rotation [in the sense determined by Eq. (1)] and causing an increase or a decrease in the moment of momentum $\mathbf{M}$. It realizes the interaction between the $\mathbf{u}$ and $\mathbf{M}$ fields.

(v) Only the constituent $\mathbf{m}_f$ of the body moment $\mathbf{m}$ is associated with external body forces. Constituents of the body moment $\mathbf{m}_1$ and $\mathbf{m}_2$, defined in Eqs. (8) and (9), are caused by the cascading scatter of moment of momentum $\mathbf{M}$ and by the mean flow modified eddy structure.

(vi) The decomposition of the total turbulence energy within the turbulence mechanics, based on Eqs. (4)–(6) into two sublevels is substantial, owing to the difference in the character of the energy interaction processes of energies K_1^t and K_2^t with energy $u^2/2$. As opposed to energy K_2^t, energy K_1^t can transform into the energy $u^2/2$. This effect, known as "negative viscosity," follows naturally from the adopted assumptions without any transgression against physical reasonability. The situation realizes when $\boldsymbol{\sigma}\cdot\boldsymbol{\omega}\geqslant\mathbf{0}$.

III. CLOSURE

Closure of the derived balance equations formulates in two steps.

The first step stands in formulation of constitutive relations. In accordance with the common rheology we assume that the generalized forces of a description depend linearly on those generalized velocities on which they act. Within this statement we have

$$\sigma_{(ij)} = -P\delta_{ij} + 2\mu u_{(i,j)}, \tag{11}$$

$$\boldsymbol{\sigma} = 4\gamma(\boldsymbol{\Omega} - \boldsymbol{\omega}), \tag{12}$$

$$m_{ij} = \vartheta_0\Omega_{k,k}\delta_{ij} + \vartheta_1\Omega_{i,j} + \vartheta_2\Omega_{j,i}, \tag{13}$$

and

$$\rho\mathbf{m}_1 = -4\kappa\boldsymbol{\Omega}. \tag{14}$$

In Eqs. (11)–(14) $P=\langle p\rangle+\frac{2}{3}K^t$; μ, γ, ϑ_0, ϑ_1, ϑ_2, and κ are coefficients, characterizing the medium properties ($\mu=\mu^m+\mu^t$,where μ^t denotes coefficient of turbulent shear viscosity; γ denotes the friction coefficient in relative rotation, i.e., if $\boldsymbol{\Omega}\neq\boldsymbol{\omega}$; ϑ_0, ϑ_1, and ϑ_2 describe diffusion of $\mathbf{M}$; κ describes decay of $\mathbf{M}$ due to the cascading scatter of moment of momentum in turbulent medium). All coefficients, included in Eqs. (11)–(14), can be, at least in principle, determined from direct measurements, based on Eqs. (11)–(14), or by comparing results of calculations with the corresponding experimental data. The constitutive relations (11)–(13) are familiar to the moment hydrodynamics [4–9], while the relation (14) expresses a property specific to the turbulent media.

Postulating inequality

$$\Psi \geqslant 0,$$

for μ, γ, ϑ_0, ϑ_1, ϑ_2, and κ we have from Eq. (10)

$$\mu,\gamma,\kappa,\vartheta_0+\tfrac{2}{3}(\vartheta_1+\vartheta_2), \quad \vartheta_1+\vartheta_2,\vartheta_1-\vartheta_2 \geqslant 0.$$

Finally, representing expression (8) for $\mathbf{m}_2$ in the form [15]

$$\mathbf{m}_2 = J(\boldsymbol{\nabla}\mathbf{u})\cdot\boldsymbol{\Omega} \tag{15}$$

(which includes the assumption about approximate linearity of $\mathbf{u}$ field within the space scales determined by R), after replacing in Eqs. (4) and (5) the quantities appearing on the left side of Eqs. (11)–(15) by their expressions on the right side of Eqs. (11)–(15), we have

$$\rho\frac{D}{Dt}\mathbf{u} = -\boldsymbol{\nabla}p + \{(\mu u_{(i,j)})_{,j}\} + \boldsymbol{\nabla}\times\gamma(2\boldsymbol{\Omega}-\boldsymbol{\nabla}\times\mathbf{u}) + \rho\mathbf{F}, \tag{16}$$

$$\rho\frac{D}{Dt}J\boldsymbol{\Omega} = \boldsymbol{\nabla}[\vartheta_0(\boldsymbol{\nabla}\cdot\boldsymbol{\Omega})] + \{(\vartheta_1\Omega_{i,j})_{,j}\} + \{(\vartheta_2\Omega_{j,i})_{,j}\} - 2\gamma(2\boldsymbol{\Omega}-\boldsymbol{\nabla}\times\mathbf{u}) - 4\kappa\boldsymbol{\Omega} + \rho J(\boldsymbol{\nabla}\mathbf{u})\cdot\boldsymbol{\Omega} + \rho\mathbf{m}_f. \tag{17}$$

The assumptions listed in the paragraph preceding Eqs. (4)–(6), constitutive relations (11)–(14), and the assumption about approximate linearity of the $\mathbf{u}$ field within the space scales determined by R, leading to Eqs. (16) and (17), form the axiomatic base of the formulated turbulence mechanics.

The second step of solving the closure problem stands in specification of μ, γ, κ, ϑ_0, ϑ_1, ϑ_2, and J. In the following we consider three sets of specifications.

a. Standard formulation of Theory of Rotationally Anisotropic Turbulence (*RAT theory*). Within RAT theory μ, γ, κ, ϑ_0, ϑ_1, ϑ_2, and J are considered as depending only on integral parameters (such as Reynolds number) of flow pattern. In this case Eqs. (16) and (17) simplify to the forms

$$\rho\frac{D}{Dt}\mathbf{u} = -\boldsymbol{\nabla}P + \mu\Delta\mathbf{u} + \gamma\boldsymbol{\nabla}\times(2\boldsymbol{\Omega}-\boldsymbol{\nabla}\times\mathbf{u}) + \rho\mathbf{F}, \tag{18}$$

$$\rho J\frac{D}{Dt}\boldsymbol{\Omega} = (\vartheta_0+\vartheta_2)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\Omega} + \vartheta_1\Delta\boldsymbol{\Omega} - 2\gamma(2\boldsymbol{\Omega}-\boldsymbol{\nabla}\times\mathbf{u}) - 4\kappa\boldsymbol{\Omega} + \rho J(\boldsymbol{\nabla}\mathbf{u})\cdot\boldsymbol{\Omega} + \rho\mathbf{m}_f. \tag{19}$$

Equations (18) and (19) differ from the equations used in Refs. [10–12] by specification of all terms through characteristics of nonaveraged flow field as well as by additional terms $-4\kappa\boldsymbol{\Omega}$ and $\rho J(\boldsymbol{\nabla}\mathbf{u})\cdot\boldsymbol{\Omega}$ on the right side of Eq. (19). Term $-4\kappa\boldsymbol{\Omega}$ describes the effect of scattering of $\mathbf{M}$ due to the cascading process and term $\rho J(\nabla \mathbf{u}) \cdot \mathbf{\Omega}$ plays an important role in establishing correspondence between the derived motion equations and classical equations. There exist two ways to achieve this correspondence. Consider the situation where $\mathbf{F}=\mathbf{0}$ and $\mathbf{m}_f=\mathbf{0}$. The first way realizes for $\gamma=0$, i.e., if there is no friction in relative rotation [see Eq. (12)]. In this case Eq. (19) declares that $\mathbf{\Omega}$, equal to zero at the initial time instant, stay equal to zero for every following time instant. The second way realizes for $\mathbf{\Omega}=\boldsymbol{\omega}$ (there is no relative rotation in a medium), $\kappa=0$ (there is no cascading scatter of $\mathbf{M}$), and $\vartheta_1=J\mu$. For the latter case Eq. (19) reduces to the equation for ω, following from Eq. (18).

b. RAT/K models. These models link RAT theory to K models. Formulation of RAT/K models is based on expression of μ^t and ψ as $\mu^t=c_1\rho\ell_2\sqrt{K_2^t}$ and $\psi=c_2\rho K_2^{t3/2}/\ell_2$ (ℓ_2 denotes characteristic length scale of turbulence constituents described by K_2^t) and on solving the problem of determination of $\mathbf{h}_2^t$ in a form reducing for the rotationally isotropic turbulence to the expression $\mathbf{h}_2^t=k\nabla K_2^t$ with $k=c_3\ell_2\sqrt{K_2^t}$ (c_1, c_2, and c_3 denote dimensionless constants). Within RAT/K models Eq. (6) for K_2^t becomes an essential component of the setup of turbulent motion description. For rotationally isotropic turbulence a RAT/K model reduces to a corresponding K model. Dependent on specification of K model used for formulating the corresponding RAT/K model and on specification of μ, γ, κ, ϑ_0, ϑ_1, ϑ_2, J, and $\mathbf{h}_2^t$, different versions of RAT/K models can be formulated.

The energy equation for K_1^t is equivalent to the equation of moment of momentum, therefore the difference in terms of the energy treatment RAT theory and K models can be formulated as follows: if K models consider total turbulence energy K^t with turbulence considered to be rotationally isotropic, then RAT theory considers only a part of the total turbulence energy associated with the two-dimensional turbulence constituent, caused by the rotational anisotropy.

c. RAT/K-ε models. These models link RAT theory and K-ε models of turbulence (in our notations $\varepsilon=\psi$). Within RAT/$K-\varepsilon$ models ψ determines from an additional equation for ψ [18]. As in the case of formulation of RAT/K models, different versions of K-ε models can be used to get different versions of RAT/K-ε models. For rotationally isotropic turbulence RAT/K-ε models reduce to the respective K-ε models used for solving the closure problem.

d. RAT/K-ω models. These models link RAT theory and K-ω models of turbulence [19] where the notion ω is used to denote the turbulent frequency. The simplest formulation of RAT/K-ω model defines ω *as* $\ell/\sqrt{K_2^t}$, leading to the following expressions for μ^t, ψ, and k: $\mu^t=c_1\rho\tau K_2^t$, $\psi=c_2\rho K_2^t/\tau$, and $k=c_3\rho\tau K_2^t$ where $\tau=\omega^{-1}$.

Besides the formulated two steps of solving the closure problem for Eqs. (4)–(6) the terms $\mathbf{F}$, $\mathbf{m}_f$, and q in Eqs. (4)–(6) must be also specified dependent on the nature of external force field $\mathbf{f}$.

IV. EXAMPLE: VELOCITY PROFILES OF ONE-DIMENSIONAL FLOWS IN PLANE CHANNEL, ROUND TUBE, AND BETWEEN ROTATING CYLINDERS

Let us consider the flows in plane channel, round tube, and between rotating cylinders within the standard formulation of RAT theory.

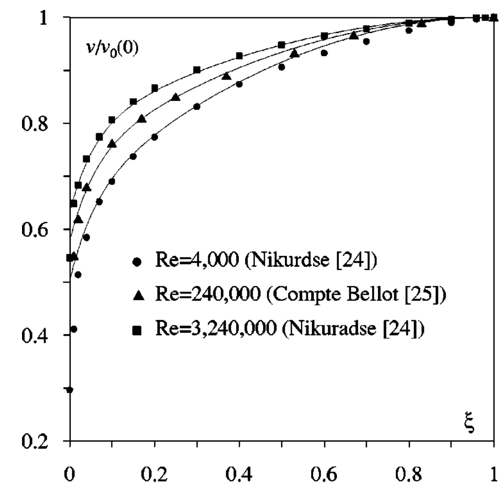

FIG. 2. Velocity profiles calculated for the Poiseuille flows in tube and channel (continuous curves), compared with experimental data of Nikuradse [20] and Compte-Bellot [21].

Applying Cartesian system of coordinates (x,y,z) in case of channel flow and cylindrical system of coordinates (r,φ,z) in case of flow in tube and between rotating cylinders, the velocity field is determined as follows: $\mathbf{u}=(0,0,u(x,t))$ for the channel flow, $\mathbf{u}=(0,0,u(r,t))$ for the tube flow, and $\mathbf{u}=(0,u(r,t),0)$ for the flow between rotating cylinders. In case of absent external force field equation set (18) and (19) simplifies to the form

$$\rho\frac{\partial}{\partial t}\mathbf{u} = -\nabla P + \mu\Delta\mathbf{u} + \gamma\nabla\times(2\mathbf{\Omega} - \nabla\times\mathbf{u}), \quad (20)$$

$$\rho J\frac{\partial}{\partial t}\mathbf{\Omega} = \vartheta_1\Delta\mathbf{\Omega} - 2\gamma(2\mathbf{\Omega} - \nabla\times\mathbf{u}) - 4\kappa\mathbf{\Omega}. \quad (21)$$

Integration of Eqs. (20) and (21) for Poiseuille flows in plane channel and in round tube leads to the following expressions for $u/u(0)$:

$$\frac{u}{u(0)} = 1 - \frac{1}{2\mu_{ef}u(0)}\left|\frac{\partial P}{\partial z}\right| H^2\xi^2 - C\frac{\cosh(H\xi/l)}{\cosh(H/l)} \quad (22)$$

and

$$\frac{u}{u(0)} = 1 - \frac{1}{4\mu_{ef}u(0)}\left|\frac{\partial P}{\partial z}\right| r_o^2\xi^2 - C\frac{I_0(r_0\xi/l)}{I_0(r_0/l)}. \quad (23)$$

In Eqs. (22) and (23) $\xi=x/H$ or r/r_0, where H is the half-width of the channel and r_o is the tube radius; $\mu_{ef}=\mu+\gamma\kappa/(\gamma+\kappa)$; $l=\sqrt{\vartheta_1(\mu+\gamma)/4[(\mu+\gamma)\kappa+\mu\gamma]}$; I_o is the modified Bessel function of zero order; and C is the integration constant.

Figure 2 presents a comparison of calculated velocity profiles with experimental data of Nikuradse [20] and Compte-Bellot [21]. The calculations correspond to $C=0.16$, $l/H=l/r_0=0.16$, and to values of μ_{ef} depending on the Reynolds number. Figure 3 presents the values of $\log(\mu_{ef}/\mu^m)$ for different log Re determined from the Nikuradse experiment (dots) and the approximating curve corresponds to μ_{ef}/μ^m

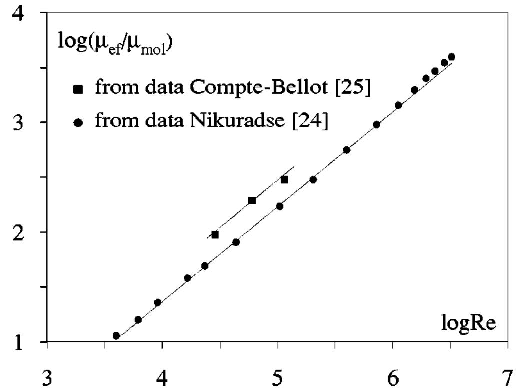

FIG. 3. The dependence $\log(\mu_{ef}/\mu_{mol})=\log(\mathrm{Re})$ determined from data of Nikuradse [20] and Compte-Bellot [21].

$=0.008\ \mathrm{Re}^{0.856}$. The same figure also shows the dependence for channel flow, as obtained from the data of Compte-Bellot. It differs from the previous example in the proportionality constant, which now is 0.014 instead of 0.008.

The solution of Eqs. (18) and (19) for the Couette flow leads to the velocity distribution, expressed as

$$\frac{u}{U}=C\xi+\left(\frac{u(1)}{U}-C\right)\frac{\sinh(H\xi/\xi)}{\sinh(H/l)}. \tag{24}$$

In Fig. 4, the calculated, according to Eq. (24), velocity profiles are compared with experimental data of Reichardt from [22]. In calculations $l=0,17H$, while the values of $u(1)/U$ and C are determined as $u(1)/U=0$, $C=0,29$ for Re=2900 and $u(1)/U=0.4$, $C=0,21$ for Re=3400.

Integration of Eqs. (15) and (19) for the flow between rotating cylinders results in the following expression for the velocity field:

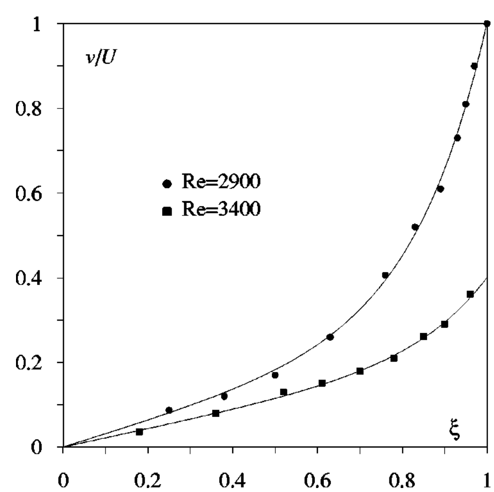

FIG. 4. Velocity profiles calculated for Couette flow in plane channel (continuous curves) compared with experimental data of Reichardt [22].

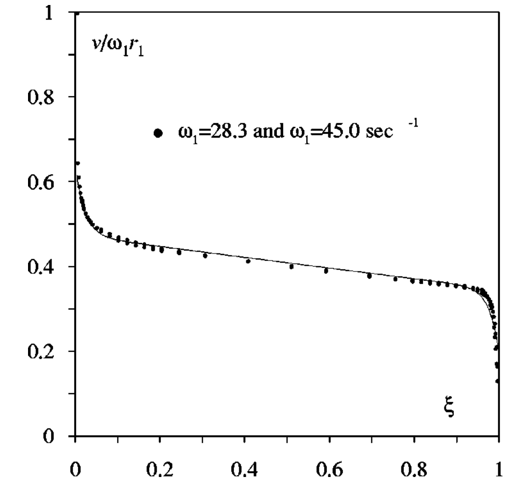

FIG. 5. Velocity profile calculated for the flow between rotating cylinders (continuous curve) compared with experimental data of Zmeikov and Ustremenko [23].

$$\frac{u}{\omega_1 r_1}=-C_1(r_1'+\xi)+C_2\frac{1}{r_1'+\xi}+C_3\frac{I_1[(r_2-r_1)\xi/l]}{I_1[(r_2-r_1)/l]}-C_3\frac{K_1[(r_2-r_1)\xi/l]}{K_1[(r_2-r_1)/l]}. \tag{25}$$

In Eq. (25) ω_1 denotes the angular rotation velocity of the inner cylinder, r_1 and r_2 are the radii of the inner and outer cylinders, $r_1'=r_1/(r_2-r_1)$, $\xi=(r-r_1)/(r_2-r_1)$, I_1 is the modified Bessel function of the first order, K_1 is the Hankel function of the first order, and $C_1,\ldots,C_4$ are integration constants. The calculated velocity profile, corresponding to $C_1=0.049$, $C_2=15.5$, $C_3=0.17$, $C_4=1.15$, is compared in Fig. 5 with data of Zmeikov and Ustremenko [23], realized for $r_1=66.6$ cm, $r_2-r_1=4.9$ cm and for $\omega_1=28.3\ \mathrm{s}^{-1}$ and $\omega_1=45.0\ \mathrm{s}^{-1}$ (the outer cylinder is resting).

The solution of Eqs. (18) and (19) for an undulating flow in a round tube realized under the pressure gradient field

$$\frac{\partial P}{\partial z}=P_0+P_1\cos\left(\frac{t}{\tau}\right), \tag{26}$$

where P_o and P_1 are constants, leads to the velocity distribution represented as

$$u(r,t)=u_0(r)+\mathrm{Re}\left[u_1(r)\exp\left(-i\frac{t}{\tau}\right)\right], \tag{27}$$

where $u_0(r)$ is the solution of the case with $\tau^{-1}=0$ determined according to Eq. (23) and u_1 expresses as

$$u_1=u_0(0)\left[-i\frac{P_1\tau}{\rho u_0(0)}+C_1\frac{I_0(\lambda R/\xi)}{I_0(\lambda R)}+C_1\frac{I_0(\lambda^* R/\xi)}{I_0(\lambda^* R)}\right]. \tag{28}$$

If $\mu_{ef}=\vartheta/J$, then for λ and λ^* we have expressions

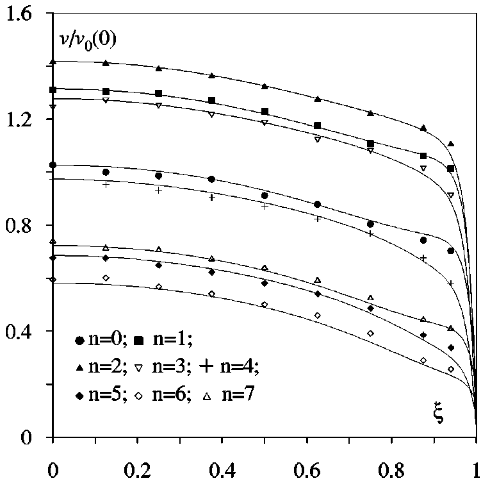

FIG. 6. Calculated velocity profiles for undulating flow in tube for different time instants (continuous curve) compared with data of Bukreejev and Shakhin [24].

$$\lambda = \frac{1}{l}\sqrt{1 - i\frac{\rho l^2}{(\mu+\gamma)\tau}} \quad \text{and} \quad \lambda^* = \frac{1-i}{l^*},$$

where $l^* = \sqrt{2\mu_{ef}\tau/\rho}$. Assuming that

$$1 \ll \frac{r_0}{l^*} \quad \text{and} \quad r_0\sqrt{\frac{\rho}{(\mu+\gamma)\tau}} \ll \frac{r_0}{l},$$

solution (28) simplifies to the form

$$\frac{u_1}{u_0(0)} = A\sin\frac{t}{\tau} + C_1'\exp\left(r_0\frac{\xi-1}{l}\right)\sin\left(\frac{t}{\tau}+\varphi_1\right) + C_2'\exp\left(r_0\frac{\xi-1}{l^*}\right)\cos\left(r_0\frac{\xi-1}{l^*}+\frac{t}{\tau}+\varphi_2\right), \tag{29}$$

where $A = P_1\tau/\rho u_0(0)$.

Figure 6 presents velocity profiles calculated according to Eqs. (23), (27), and (29), for time instants $t/\tau = n\pi/4 + 0.02\pi$ where $n = 0, 1, \ldots, 7$ and $\tau = (20\pi)^{-1}$ s. The calculated profiles are juxtaposed with the data obtained by Bukreejev and Shakhin [24]. The calculation parameters are determined as $P_0/4\mu_{ef}u(0) = 0.37$, $C = 0.5$, $A = -C_1' = 0.419$, $C_2' = 0.081$, $\varphi_1 = \varphi_2 = 0$, and $l^*/r_0 = 0.15$.

Besides the presented velocity distributions, the distributions of many other quantities, such as characteristics of internal rotation and of the stress and energy, can be calculated.

V. CONCLUDING REMARKS

The only reason for generating a nonzero **M** field, defined by Eq. (1), follows from the eddy structure of the turbulent medium. Generating a nonzero **M** field does not presume any micromorphic properties of the continuum [10–12] or finiteness of the linear scale of differential volume [13]. This assertion is valid for the turbulence treatment formulated in the present paper as a whole.

The formulated turbulence mechanics essentially widens the physical background of the turbulence mechanics as well as enlarges its capacity to describe various effects (like effects of the so-called "negative viscosity"). It opens the door for generalization of different models (such a K models, K-ϵ models, and K-ω models) used in a variety of applications.

The formulated turbulence mechanics offers not only an additional instrument for discussion of different theoretical turbulence problems but, as it is shown in Sec. IV, also an useful tool for practical calculations.

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